Journal for Research in Mathematics Education
2012, Vol. 43, No. 5, 616–650

Making Connections Among Student Learning, Content, and Teaching: Teacher Talk Paths in Elementary Mathematics Lesson Study

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This study investigated how elementary teachers in a mathematics lesson study made sense of student learning, teaching, and content, as related to using representations in teaching multidigit subtraction, and how changes occurred over time in their talk and practice. The lesson-study process paved a group talk path along which teacher talk shifted from superficial to deeper consideration of student learning. By providing a context in which interactions of diverse ideas drove teacher learning, lesson study facilitated teachers making connections between the craft knowledge of teaching and scholarly knowledge. Individual teacher talk paths varied within the group path, and one teacher's learning path and the interaction of different learning paths is discussed.

Key words: Addition, subtraction; Elementary, K-8; In-service teacher education; Problem solving; Professional development; Qualitative methods; Representations, modeling; Teaching practice

Informed by the new vision of mathematics teaching described by the National Council of Teachers of Mathematics (NCTM, 1989, 2000), many teachers at different professional stages attempt to teach for meaningful understanding, but

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This study was supported by the Wallenberg Global Learning Network II Preparation and Full grants. The authors thank the teachers and students who participated in the study. We also acknowledge the valuable feedback provided by close and distant colleagues on previous versions of the manuscript as we produced multiple revisions.

find it difficult to do so (Ball, 1990; Brown, Cooney, & Jones, 1990; Ebby, 2000). Most people, including many teachers, learned mathematics in their youth by mastering procedures, and they struggle to make sense of more conceptual and inquiry-based teaching. They may adopt only surface features of teaching for deep understanding and thus find it difficult to promote student learning (Cohen, 1990; Chazan & Ball, 1999), especially without well-structured and targeted professional guidance. It is clear that helping teachers to improve mathematics pedagogy[2] in classrooms is a key to creating better learning opportunities for students (Darling-Hammond, 2001).

Research on teacher professional development has demonstrated that teachers are more likely to change their thinking and practice when they have opportunities to learn about student thinking about mathematics and discuss their instruction for student thinking about mathematics (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Toward that end, collaboration-based, practice-oriented teacher professional development has been found to be more effective than externally administered programs such as one-time lectures and single-day workshops (Borko, 2004; Borko, Davinroy, Bliem, & Cumbo, 2000; Borko & Putnam, 1995; Kennedy, 1998; Wineburg & Grossman, 1998). This article describes a case of teacher learning in one such professional development program centered on elementary mathematics lesson study.

By following the shifts in teacher talk during a yearlong lesson-study cycle, the study aims to understand, at a detailed level, the teaching and teacher development that result from connecting scholarly knowledge to the craft knowledge of teaching[3] (Ruthven & Goodchild, 2008). Following a brief summary of relevant literature and the research methods is a discussion of how the teacher talk in the lesson-study group shifted over several meetings as the teachers increasingly focused on evidence and made stronger connections among the content, student understanding, and their teaching. We will then discuss how these shifts were reflected in one teacher's mathematics pedagogy.

LESSON STUDY

Lesson study originated in Japan, and it supported a shift in teaching practices, from didactic to more student-centered, from the 1980s to 1990s (Lewis & Tsuchida, 1998; Murata & Takahashi, 2002). It is a professional development strategy that helps teachers explore effective teaching practices (Chokshi & Fernandez, 2004; Lewis, Perry, & Hurd, 2004; Lewis, Perry, & Murata, 2006) and

[2]By mathematics pedagogy, we mean the ways in which teachers teach mathematics, such as the questions they ask, the participation structures they use, and the ways in which they promote and extend students' ideas.

[3]Ruthven and Goodchild (2008) define craft knowledge of teaching as that which is created within the practice of teaching, while scholarly knowledge of teaching is created within the practice of re-searching.

has gathered global attention in the last decade (Fujita, Hashimoto, & Hodgson, 2004; Lo, 2003; Murata, 2010, 2011). As a model of professional development, lesson study provides a context for teachers to expand their knowledge for teaching (Fernandez, 2005; Yoshida, 2008) through an organized cycle of (a) goal setting, (b) curriculum analysis, (c) lesson planning, (d) teaching a lesson while being observed, and (e) debriefing and reflecting, in an open and collaborative setting.

Although some teachers may initially lack the mathematical content knowledge and research dispositions that are important in lesson study, these qualities can be developed gradually through participating in the lesson-study process (Fernandez, 2005; Fernandez, Cannon, & Chokshi, 2003). Lesson study provides multiple opportunities for teachers to experience their everyday practice with focus and depth over a longer time period (Murata, 2010, 2011).

Problem Representations and Solution Methods

Aside from providing teachers with an avenue to develop their knowledge, lesson study provides a context for teachers to explore the effectiveness of many different practices (Chokshi & Fernandez, 2004). The lesson-study process described in this article involves a group of U.S. elementary mathematics teachers who explored the uses of a visual representation (number lines) in multidigit subtraction problem situations. This topic was chosen by the teachers as they wanted to learn how to use representations to connect different student solution methods with subtraction situations. By the time students are in mid-elementary grades, they routinely use algorithms for multidigit operations (e. g., subtraction), but they may lack a firm understanding of place value (Carpenter, Franke, Jacobs, Fennema, and Empson, 1998; Fuson, 1992; Kamii, 1985). Without understanding place value, the carrying and borrowing of numbers across places with an algorithm can be a meaningless procedure; thus, students cannot find and correct their own errors by tying the procedure back to the original problem (Brown & VanLehn, 1982; Carpenter et al., 1998).

The National Council of Teachers of Mathematics describes *representation* as "the act of capturing a mathematical concept or relationship" (NCTM, 2000, p. 67). Representations play important roles in helping communicate mathematics problem situations, students' ideas, and solution methods. For example, students may use base-ten blocks to model quantities in terms of their place values (e.g., 10s and 1s). However, these hands-on experiences may not be connected to their learning of multidigit subtraction or their use of algorithms (Cohen, 1990). Murata (2008) reports how visual representations are used differently in U.S. and Japanese textbooks and suggests that U.S. teachers may use them sporadically and randomly, possibly keeping students from developing conceptual familiarity with why and how to use certain representations for particular problems.

Duval (2006) argues that semiotic representations must necessarily be used for any mathematical activity that encompasses mathematical objects, because the mathematical objects can be accessed only through representations. In teaching mathematics, however, it is possible to put aside mathematical objects and representations of the objects, and instruction can focus on procedures at each grade level. In this way, students typically do not have enough experience with analyzing and using representations to understand the unique characteristics of the objects being represented. For example, in the context of the model method (a visual representation used in Singaporean mathematics curriculum), the model itself is a problem-solving heuristic which, to be used as such, requires students to reflect on how they can accurately represent the information in the problem (Ng & Lee, 2009). Instead, students may use the model method as an algorithm. Ng and Lee (2009) call the model method "the art" (p. 312) of representation, a method that needs to be taught to students initially to enable it to become a tool for students to use later to solve mathematics problems. Although the distinction between representations of mathematical situations and representations of solution methods may be unclear at times in classroom instruction, it is important for students and teachers to understand the differences between them and find ways to use representations that show the situation and lead to problem solution (Murata & Kattubadi, 2012).

These ideas guided the design of the study reported as we focused our analysis on how teachers talked about the uses of representations and how they used them in their teaching. We wanted to know how the teachers in the study might or might not distinguish between representations of situations and representations of mathematical solutions in their talks and/or their teaching, and how these distinctions might develop or change over time in their lesson-study collaboration and in actual teaching. By revealing these processes, we hoped the study would shed some light on teacher learning and development.

CONCEPTUAL FRAMEWORK

As claimed previously, teachers need to learn to teach their students better, and in order to support teacher learning, teaching needs to be better understood. In conceptualizing teaching, prior research used a three-way model similar to the model illustrated in Figure 1 (e.g., Cohen and Ball, 1999; Lampert, 2001; National Research Council, 2001).

In *Adding It Up: Helping Children Learn Mathematics*, the National Research Council (2001) describes teaching as consisting of interactions between teachers and students around content. As described in the framework reported in *Adding It Up,* teachers bring their knowledge, beliefs, decisions, and actions to the context in which they interpret their students' needs and facilitate students' experiences with the content. The authors of *Adding It Up* discuss how students bring their experiences and understanding and simultaneously make their own decisions using the opportunities presented for them to learn. Cohen and Ball (1999) also assert that the interactions among the three key dimensions shown in Figure 1 frame student learning.

Lampert (2001) further explains that teachers' understanding of the relationships among the three dimensions develops in classroom interactions, and therefore it is classroom-specific. Teachers' knowledge of students influences how teachers facilitate the interactions between students and the content, and there is great

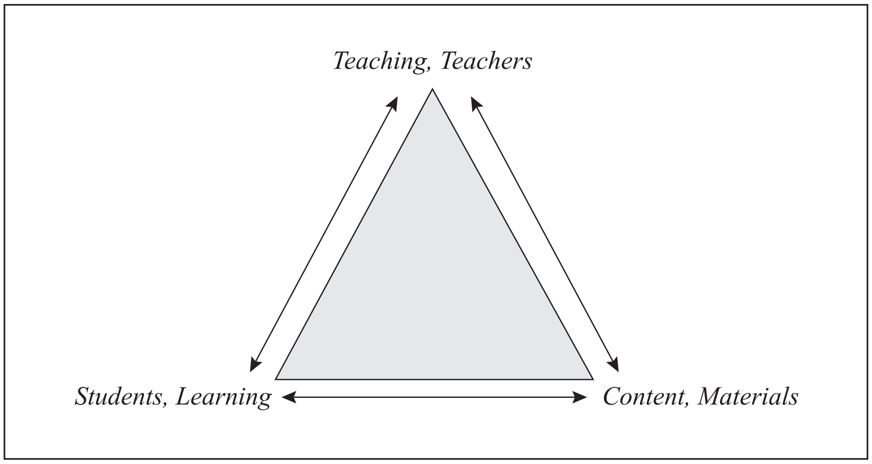


Figure 1. Conceptual framework: The teaching–students–content triangle.

variability in how the three-way interactions unfold from one classroom to the next, even when the same curriculum with predetermined content is used. If interactions among the three dimensions characterize teaching, then student learning outcomes depend largely upon how teachers understand and facilitate such interactions as they happen. This was the starting point of the current study; specifically, the following are the research questions addressed:

- In the context of lesson study in which they engage, how do teachers make sense of student learning, teaching, and content, as related to the uses of number-line representations with multidigit subtraction?
- How do their foci on each of these dimensions change over time?

METHOD

Participants and Setting

Three elementary school teachers participated in the lesson study. The teachers taught Grades 2, 3, and 4, respectively, in a Grades K–5 public elementary school in the western United States. The city in which the school is located has mixed-income socioeconomic status, with 24% of students at the school belonging to socioeconomically disadvantaged households and 32% of students designated as English Language Learners and receiving special instruction at the school. The teachers had varied amounts of teaching experience at the time of the study. The Grade 2 teacher, Rosemary,[4] had taught for 13 years; the Grade 3 teacher, Andrea, had taught for 2 years; and the Grade 4 teacher, Sophia, had taught for 16 years. The principal, Amy, also often joined the lesson-study meetings. She was a former elementary and mathematics (high school and college level) teacher and had been the principal of the school for 10 years.

[4]All names used are pseudonyms.

The first author of this article joined the lesson-study group as a facilitator to focus the teachers' work and guide their thinking. She created a lesson-study agenda at the beginning of the year with plans for each meeting (discussed subsequently), and helped focus teachers' attention on the key topics in the meetings. She often started the meetings by summarizing what had been discussed previously and proposing what to accomplish in the meeting, and ended the meetings by relaying what the teachers could anticipate for the next meeting. She also provided content support by answering teachers' questions about the mathematics topics and asked questions to further probe their thinking about the topics. When appropriate, she also provided research literature and curriculum resources for teachers to use as they planned lessons (e.g., when teachers had a question about students' typical learning trajectories or asked how the same content was treated in different textbook materials).

The school used two mathematics textbook curricula: *Investigations in Number, Data, and Space* (Pearson Education, 2008) and *California Math* (Houghton Mifflin, 2006) and also provided teachers with a pacing guide that outlined how to combine and integrate these curricula to meet all state-specified content standards for each grade level. Teachers also had the freedom to supplement and modify the given curricula to fit their students' needs when appropriate, and they often collaborated in grade-level teams to support each other's work at the school.

Prior to the year of the lesson study, the teachers and the first author met and discussed possible lesson-study topics. Considering the multigrade nature of the group, the teachers wanted to find common topics that spanned the grades and would be challenging to teach and learn. They decided on multidigit subtraction and place value as focus topics. The first author then visited individual classrooms and conducted clinical interviews with students of that year to understand the general patterns and trends of their strategies and thinking about the topics. The results of the interviews were compiled, summarized, and shared with the teachers at the beginning of the lesson study to provide baseline information on student learning.

The teachers and researchers met a total of eight times over 6 months of the school year for lesson study. Their meetings were held after school hours, approximately once per month, and each lesson-study meeting lasted 60 to 90 minutes. Table 1 presents an overview of the general activities for each meeting. The research lessons were all taught on the same day—the day on which Meeting 8 occurred in the afternoon. Each teacher taught a modified form of the same lesson while being observed by all the other teachers.

Lesson Study Overview

In starting lesson study, the teachers agreed that multidigit subtraction and place value were challenging mathematics topics for their students to learn and difficult to teach. In Meeting 1, the results from the previous year's student assessment were reviewed, and the following observations were discussed. Grade 2 and Grade 3 students used various methods to solve multidigit subtraction problems by

Table 1
Lesson-Study Meetings and Main Activities

Meeting	Main activities
1	Discussing student data from the previous year • Sharing the summary of the student interview results • Grade 2 teacher sharing her experiences from the subtraction lesson she taught that day and examining and discussing student methods (this provided concrete data to analyze) • Discussing the development of student methods across grade levels • Sharing each other's grade-specific teaching knowledge
2	Discussing student work from teachers' classrooms • Sharing and discussing teaching in individual classrooms on subtraction • Discussing typical challenges they saw in student methods, regarding place value and the algorithm • Discussing different representations (tape diagrams, number lines, tally marks) of subtraction problems • Discussing approaches different curricula take to teach subtraction
3	Planning Meeting 1: Examining curriculum materials • Each teacher discussing the lesson plan she brought for the meeting, including general lesson structure, problems, and representations used • Discussing different problems and questions and how they elicit certain student methods and representations • Discussing how each lesson could be modified for different grade levels • Identifying the goals of the lesson • Relating subtraction problems by showing them with different representations (number line, tape diagram, algorithm)
4	Planning Meeting 2: Examining curriculum materials • Analyzing the lesson plan regarding teacher prompts, student methods, and representations • Discussing the focus of the lesson and what teachers are interested in learning in the lesson-study process • Deciding on the lesson and deciding on how to modify lesson plans across grade levels
5	Planning Meeting 3: Discussing lessons • Discussing each problem in the lessons in detail regarding problem types, numbers used, how they may modify and present problems • Discussing teacher prompts in the lesson and how to help students use number-line representations • Discussing the logistics of the lesson day

5	Planning Meeting 3: Discussing lessons • Discussing each problem in the lessons in detail regarding problem types, numbers used, how they may modify and present problems • Discussing teacher prompts in the lesson and how to help students use number-line representations • Discussing the logistics of the lesson day
6	Planning Meeting 4: Finalizing lesson plans • Each teacher discussing her lesson plan, asking questions, and receiving feedback on the details of the plans (e.g., the order of the problems, numbers to use, activity structure, questioning) • Brainstorming possible student responses to different problems • Solving problems on their own, representing them using number lines, and realizing different ways to represent
7	Planning Meeting 5: Discussing details and fine-tuning lessons • Discussing the research lesson day regarding timings, processes, and logistics • Discussing lesson process regarding grouping of students and timing of the different parts of the lesson • Grade 3 teacher sharing her teaching from the day, focusing on student thinking behind subtraction methods
8	Research lesson debriefing • Discussing student learning in the lessons observed (based on data collected) • Discussing goals and how they assessed student learning of the goals in the lesson

Note. Unless specified otherwise, all three teachers, the principal, and the facilitator participated in the activities.

decomposing numbers and working within matching place values (e.g., first subtracting the numbers in the 10s place, then the numbers in the 1s place). These younger students typically made errors when they could not keep track of the decomposed numbers with which they were working in the process. For example, students may approach 63 – 27 by decomposing 63 to 60 and 3, 27 to 20 and 7, and trying to subtract 60 – 20 and 3 – 7, realizing that 7 is larger than 3 in the 1s place, decomposing the numbers again to think of the two subtraction equations as 50 – 20 = 30 and 13 – 7 = 6, then performing another subtraction instead of addition, finding the incorrect answer by 30 – 6 = 24.[5] By Grade 4, almost all students used the standard algorithm, and the only error they made was switching the minuend and subtrahend in the common vertical algorithm. Statements from each of the three teachers indicated that they saw the connections between different student methods and wanted to find a way to help students in Grades 2 and 3 approach problems in more structured ways while also helping Grade 4 students use algorithms with more conceptual understanding.

As the meetings progressed, the teachers discussed different lesson possibilities to achieve these emerging goals. They also wished to find a common way to teach the topics to students across grade levels so that students would make connections easily when they moved from one grade level to the next. Some of the teachers indicated that they had previously experimented with various representations and felt successful. Using visual representations in their teaching emerged as a possible way to foster connections between students' thinking about problems and the algorithmic processes. Sophia, the fourth-grade teacher, brought a unit plan from one of the curricula the school used—*Investigations in Number, Data, and Space* (Pearson Education, 2008)—that focused on representing student thinking using number-line representations (see Appendix A). The teachers discussed the unit and decided to modify the first two lessons for three different grade levels by changing the numbers and problem context slightly.

For the research lessons, the main goal of the instructional unit narrowed from understanding the general topic of multidigit subtraction and place value to developing the knowledge to use number-line representations to bridge students' ideas in solving multidigit subtraction problems. The underlying goal remained the same (to better understand how students learned the topics and how to teach them), but the teachers generated more specific goals for their learning. It is typical for lesson-study groups to narrow their goals over time, as they come to understand the complexity of the content through collaborative investigation (Murata & Pothen, 2011), and this group was no exception.

The lesson-study format varies substantially in Japanese education, but outside of Japan, there has been primarily one format of lesson study introduced: school-based, grade level-specific, small-scale lesson study in mathematics (Murata & Takahashi, 2002). For the current study, the teachers designed and modified their own lesson study with the help of the facilitator. As a consequence, each teacher planned a different instructional unit for her own grade level, received feedback from the other teachers, and taught public lessons while being observed by all the other teachers. Although this may seem different from lesson study commonly seen in the United States, it incorporated critical components of lesson study and followed the lesson-study process of setting goals, examining curriculum materials, planning a lesson, teaching a lesson while being observed, and debriefing.

Data Collection

This article reports findings from the 2nd year of the 2-year study, focusing on answering the aforementioned research questions. For the 1st year of the study,

[5]This is a hypothetical erroneous student solution.

data were collected on elementary school students' understanding of multidigit subtraction and place value through student interviews, teacher interviews, and content analysis of curriculum materials. For this article, we used the following data collected in Year 2 of the study: videos of teacher meetings (transcribed), lesson-related materials (e.g., lesson plans) in the curricula that the teachers used or ones they created, lesson videos, student work (e.g., completed worksheets), and teacher interview notes (collected after research lessons).

Data Analysis

Teacher meeting transcripts were first reorganized into to-be-coded sentence chunks, each containing a subject and a verb. For example, "My students used a number line" is considered one chunk, and "I saw different student strategies, and it was hard to facilitate the discussion," is considered two sentence chunks as it includes two subject–verb sections. We decided to use sentence chunks because individual phrases did not always express complete thoughts that could be coded reliably. Five authors of this paper coded the transcripts in three different stages, making the coding more specific to our research questions over time. Parts of the transcripts were not coded: transcripts of talk that did not express an idea (e.g., repeating a word after someone else's statement) and transcripts of talk that was unrelated to the lesson study (e.g., discussing a scheduled school event).

The researchers coded the transcripts to examine how teachers talked about the three dimensions of the framework (teaching, learning, and content) in relation to representation uses (this was a primary lesson-study goal). From all sentence chunks, we identified those in which teachers talked about use of representations (this was 39% of the total number of transcript chunks). It was reasonable to focus on the use of representations because the use of the representations was the main goal of the lesson-study teachers identified, and we wanted to follow their learning concerning use of the representations. The smaller amount of data on which we consequently focused enabled us to work more efficiently. We then categorized each chunk as involving talk about student learning, content, or teaching. We wanted to examine the patterns in the data, to see whether teacher talk focused on some dimensions more than others in particular meetings, and whether there was a flow of the focus on the dimensions with the movement across time and patterns of change over lesson-study meetings and among different teachers.

We met several times to generate specific codes for the final analysis. We further identified different ways teachers talked about representations within the dimensions of student learning, teaching, and content. For example, for student learning, teachers sometimes talked about what representation students used, and at other times, why they used them. These differences were identified (and later placed in different levels). After generating these codes, we calculated inter-rater reliability by coding separate sentence chunks as well as transcript segments (approximately 25% of the entire set of sentence chunks), with 87% agreement (calculated as the number of sentence chunks matched divided by the total number of chunks coded). We also came to total agreement after discussing each coding on which we initially disagreed. The first author then used the final codes (see Figure 2) to code all chunks with the HyperResearch software.

As the coding process continued, we agreed that some teacher-talk codes showed deeper teacher thinking than others. Teachers sometimes talked about their teaching at more superficial levels than at others, and we noticed that there seemed to be different levels of depth corresponding to the codes we were using. For example, we coded a sentence chunk as "what teachers do" when they simply stated what they did or were planning to do in the lesson (e.g., "I am going to use this representation"), compared with "teachers' goals of the lesson" when their talk chunk stated the purpose of what they do (e.g., "Facilitation is important to connect student strategies in the lesson"). We considered these differences in teachers' conversations to be important, as they suggested different understandings of student learning, content, and teaching. We also noticed that teachers sometimes stated their ideas with or without supporting evidence, and because we encouraged teachers to discuss their practice using evidence from classrooms, we wanted to recognize when they used evidence.

After discussing these differences among codes, we identified levels for the codes and examined the progression of teacher talk over time based on those levels. We placed statements about single dimensions (by *dimensions* we mean student learning, content, or teaching, as in Figure 1) that were not accompanied by interpretations (e.g., description of what it is and how things are, without inferring why) as the lowest level (Level 1), interpretations of action or text without evidence as the next level (Level 2), interpretations with evidence as the next level (Level 3), then at the highest level, connections made across different dimensions (Level 4). We saw these levels as enabling us to address our research questions by examining such connections (see Figure 1). All codes are shown in Figure 2.

Although teacher meeting videos and transcripts were the primary data reported in this article, other data also were analyzed to support the primary findings. Lesson materials (e.g., lesson plans) and video transcripts of all research lessons were reviewed for evidence of teacher learning identified in the main analysis (teacher-talk analysis). For example, when meeting transcripts indicated how a teacher was making a connection between a student strategy and a visual representation, the lesson video was reviewed to see how a segment of the lesson worked in the actual classroom. Special attention was paid to excerpts of the teacher meeting data that indicated that teachers were making (or failing to make) connections between or among student learning, mathematics problems, and representation use.

RESULTS AND DISCUSSION

In this section, we present and discuss the results of the analysis in three parts: (a) changes in talk for the group and changes in talk for the individual teachers, (b) teacher learning in contexts (with examples of changes in teacher talk), and (c) Andrea's learning during the process.

Level	General description of the level	S/C/T *	Code names	Description of the code (when the sentence chunk described...)	Examples
1	Single-dimensional description and interpretation of action and/or text	S	What students do with representations	Concrete student action	"Some of them were using the number line."
		C	Describing curricula about representation	Concrete use of representations in a curriculum	"They (the curriculum) actually teach the number line in this set of lessons."
		T	What teachers do with representations	Concrete teacher action	"I drew the representation right on the paper."
2	Non-evidence-based interpretation of action and/or text	S	What students know about and expect with representations	Student understanding and expectations (that guide their representation use)	"And somehow they think that they have to do it that way."
		C	Affordances and limitations of representations	Certain affordances and limitations a representation offers in a curriculum or a lesson	"You can take 319 from 634 and still use the number-line representation, too."
		T	Teacher expectation and feelings about representation uses	Teacher expectations that guide their representation use	"I was really surprised at how many of them just drew it out."
3	Evidence-based interpretation of action and/or text	S	Interpreting actual student work with representations	Student understanding and expectation, based on evidence	"So what is this? Is this supposed to be books?"
		C	Possible representations and development of representations	Representation use and development in a curriculum or a lesson, based on evidence	"You can break it down or you can use the number line (as shown in the lesson plan)."
		T	Teacher understanding of representations	Teachers' understanding of representation based on evidence	"So the minuend is the bottom, that's the total the bar represents."
4	Connections made across different areas based on evidence	S	Connecting student thinking and representations	Connection between certain student thinking and particular representation use	"And they'd have to jump over 10—in their representation."
		C	Connecting problems and representations	Connection between a certain problem and particular representation use	"This kind of problem invites students to use a linear representation nicely."
		T	Teacher goals of the lesson in terms of representation uses	Connection between teachers' goals and student understanding	"Students came up with different representations, as we hoped."

* Indicates the code is about either student learning, content, or teaching.

Figure 2. Descriptions of levels of talk for student learning, content, and teaching.

Group and Individual Teacher Talk Shifts

Lesson-study teacher-talk path: Interactions among student learning, teaching, and content. Figure 3 shows the percentages of the types of talk that occurred during each meeting (column) based on the total number of sentence chunks in that meeting. Across the rows, shaded cells in the figure identify the meetings during which the higher percentages for each category occurred. Cells were shaded for the two highest percentages for each category row, except when the value was less than 10%. The highest percentages from each code (row) were highlighted, as opposed to those in each meeting (column), to bring more specific patterns about changes in teacher talk to the surface. This was done to avoid dominant categories being shaded for most meetings, such as the "what teachers do" category that was the highest occurring category for many meetings. The dashed arrow was superimposed to show the group movement of the focus of the teacher talk over time.

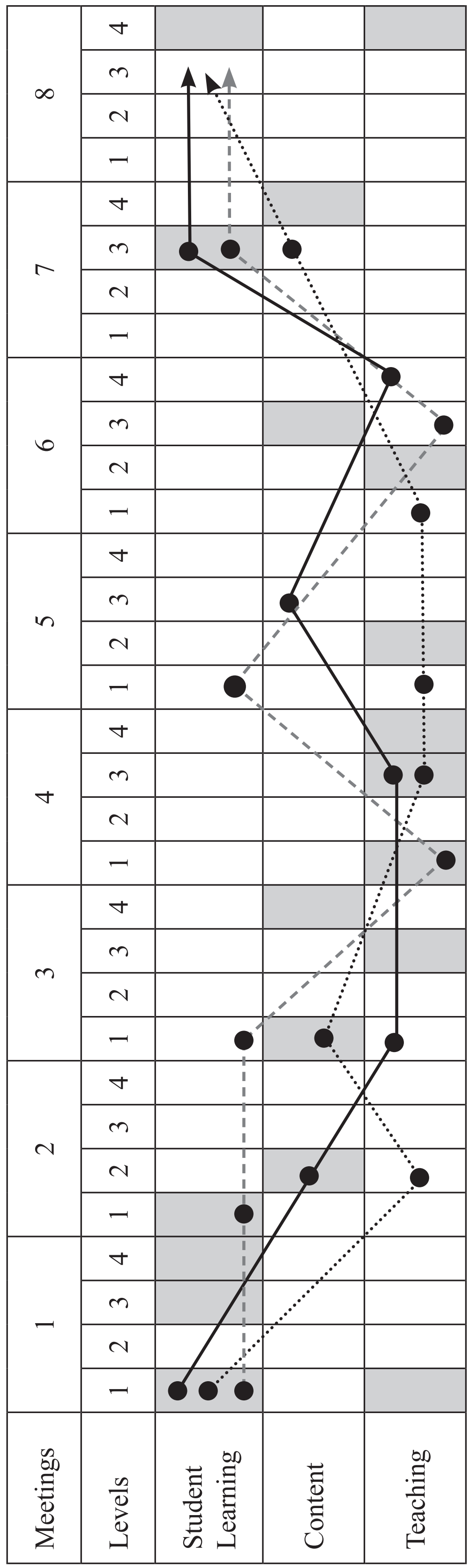
Teacher talk overall moved from discussing student learning of mathematics, to discussing content, to discussing teaching, and finally back to discussing student learning (the dashed arrow in Figure 3 shows the general pattern of movement). This pattern of teacher talk reasonably follows the trajectory of teacher talk expected in lesson study: They discussed student learning of the content at the beginning, examined the content, planned their instruction, and reflected on student learning at the end. Their discussion was facilitated by a lesson-study facilitator and followed the lesson-study cycle; thus, the talk path indicates the trajectory specific to this lesson study and was not merely teachers' spontaneous talk. This suggests that the professional development structure can explicitly lead teachers to follow certain paths, in order to connect different aspects of teaching in the process. For the current case, the group talk began and ended with student learning, but it was not a straight path, nor was their talk similar at the beginning and at the end (individual teacher-talk paths differed slightly from the group path).

Interactions among individual teacher-talk paths. Figure 4 shows individual teacher-talk paths. Each arrow path shows the dominant level and dimension of teacher talk for each teacher during each meeting. The level and dimension in which teachers' talk was concentrated is depicted by shading of the cells (as shown in Figure 3).

In the paths of talk indicated by shaded cells, individual teachers took slightly different paths. While the group-talk path was supported by the lesson study, each teacher's movement through the process shows the topic of her interest or how she perceived the topic as relevant to her thinking in a particular meeting. For example, Sophia showed movement between content and teaching, and her talk in meetings often focused on how to make content accessible through her teaching methods. In contrast, Rosemary's path remained longer in the area of teaching, showing she was concerned about her teaching more than about the other dimensions. Andrea shifted back and forth between student learning and teaching, suggesting that she was trying to understand how her teaching supported student thinking of mathematics. In discussing different teacher-learning paths in video-club meetings, van Es and Sherin

	Meetings	1	2	3	4	5	6	7	8
STUDENTS	1. What students do with representations	28%	28%	4%	10%	14%	3%	0	19%
	2. What students know about and expect with representations	0	2%	2%	2%	0	5%	3%	8%
	3. Interpreting actual student work with representations	19%	9%	0	3%	5%	0	45%	15%
	4. Connecting student thinking and representations	14%	1%	7%	4%	11%	5%	1%	15%
CONTENT	1. Describing curricula about representations	0	3%	17%	3%	7%	0	0	0
	2. Affordances and limitations of representations	7%	22%	2%	0	4%	<1%	2%	3%
	3. Possible representations and development of representations	0	5%	7%	0	9%	12%	2%	3%
	4. Connecting problems and representations	0	9%	18%	12%	8%	16%	25%	5%
TEACHERS	1. What teachers do with representations	23%	14%	19%	31%	20%	22%	13%	8%
	2. Teacher expectation and feelings about representation uses	9%	8%	8%	0	11%	12%	0	4%
	3. Teacher understanding of representations	0	1%	15%	16%	3%	8%	0	8%
	4. Teacher goals of the lesson in terms of representation uses	0	0	2%	15%	5%	14%	10%	15%

Figure 3. Percents of teacher talk falling in each category for eight lesson-study meetings.



Note. Sophia: —— Rosemary: • • • • • • Andrea: – – –

Figure 4. Individual teacher talk paths through dimensions and levels for eight lesson-study meetings.

(2008) identify key factors that contributed to the difference: teachers' knowledge, beliefs, and experiences. In our study, the teachers were also different from one another, which was evident in different foci they showed in each meeting and thus in the different trajectories they followed across meetings (see Figure 4).

In some meetings (Meetings 1, 4, 6, and 8), all three teachers' talk focused on the same dimension. In other meetings (Meetings 2 and 5), their talk split across different dimensions, with one or two teachers' talk located outside the lesson-study path. At other times, such as in Meeting 7, their talk split between two dimensions that were embedded in the group path. These differences created different kinds of conversation among teachers in the meetings. When their talk focused on a single dimension, the teachers' conversation built on each others' ideas on that topic. For example, in Meeting 6, when teachers discussed the details of the upcoming lessons, their talk focused on teaching, as it was necessary to fine-tune different aspects of the lessons. On the other hand, when their talk split among dimensions, they exchanged different ideas and added richness to the conversations—for instance, when the teachers examined different student work and shared their teaching ideas in Meeting 2 (examples follow).

Teacher-talk levels change. Figure 5 shows the changes in levels of teacher talk over eight meetings. Over time, teacher talk moved from lower to higher levels, thus becoming more evidence-based and connecting different dimensions at a greater rate.

At the beginning, teacher talk was primarily at Level 1 (> 50%). Higher level talk gradually increased over time, with some fluctuations. In Meeting 8 (the final meeting), teacher talk focused on student data collected in the research lessons, thus Level 1 teacher talk again occurred more than in previous meetings. However, one large difference between this Level 1 talk and that in the early meetings was that it was often accompanied by statements of lesson goals or representation uses, or drew from specific examples of student learning, and thus was more purposeful and connected.

Teacher Learning in Context

How did the teachers make connections among the three dimensions of the model (see Figure 1) in the process? When teachers had different foci, how did they support one another in meetings? In the following section, we illustrate, with examples, how individual teacher talk interacted to support teachers' connection-building in lesson study.

We specifically focus on one of the teachers, Andrea, to show how her ideas (indicated by her talk) were supported and/or extended in lesson-study meetings. By focusing on one teacher, we highlight her learning more purposefully with examples. We chose Andrea for several reasons. First, Andrea was the least experienced teacher of the three who participated in the lesson study, and she was most open and willing to learn. She was also open to data being collected during the

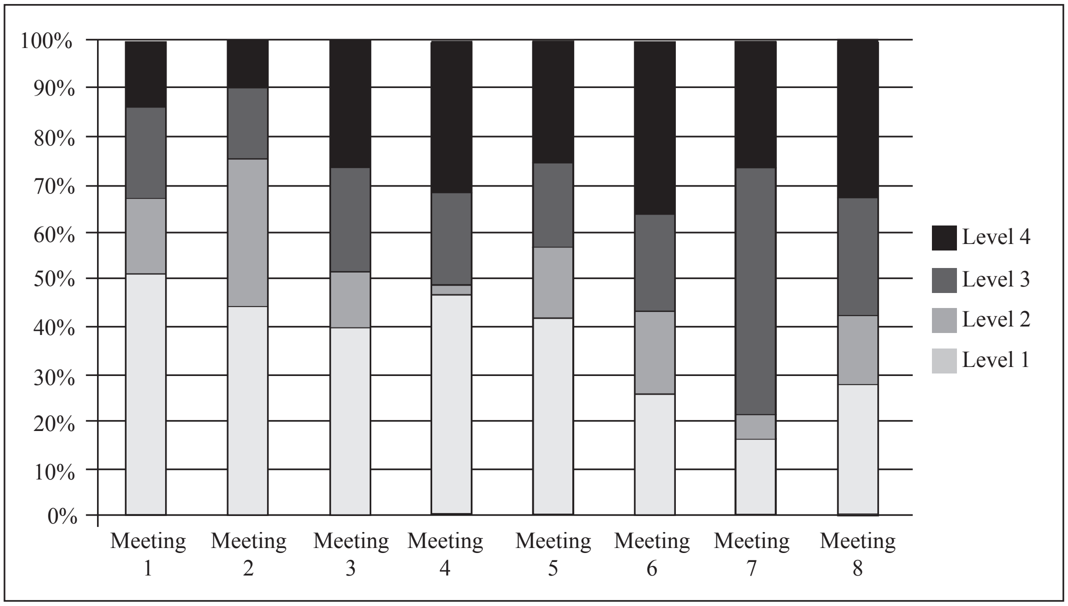


Figure 5. Percent of teacher talk at each level over eight meetings.

entire instructional unit. The other teachers opened their classrooms only for the research lessons (public lessons as a part of lesson study) for observation. Although this was sufficient to investigate the lesson-study process, more comprehensive data collection in Andrea's classroom made the investigation of her learning possible for this article.

Andrea came to the lesson study having just finished her 2nd year of teaching. Interview data show that although she felt strongly about student-centered teaching, she also felt overwhelmed at times by students' varied ideas and lacked a clear strategy about how to facilitate their ideas in discussion.

Teacher talk at the beginning of lesson study. In Meeting 2, each teacher brought data on student thinking about multidigit subtraction and place value from her classroom (this was requested by the facilitator). As each teacher shared her data, she did not build on the previous teacher's comments. This resulted in a disconnected conversation. The following excerpt shows how one teacher talked about her ideas in the meeting.

Andrea: . . . 32 minus 17, and they had to show it two different ways. They used a lot of illustrations.

Amy: And nothing in between [using algorithms and drawings]. And mostly not organized. . . . Not grouped by 10s, just 1s and . . .

Andrea: Yeah, I think Eugene was the only one who did it in groups of 10. . . . So then we had a discussion and they shared out answers. So then we spent . . . I may have spent too long . . . I'm trying to get them to think of better ways of organizing it. So, anyway, I had them show different ways. But there were just so many odd ways of doing it.

Aki: [looking at student work] This is an addition problem, 32 + 17 . . . . No matter which way they recompose the two numbers, still adding . . . [for subtraction], some parts can be subtracted, some parts can be added, and it's easy to misinterpret that.

Andrea: Yeah, that's what's tricky about these methods of breaking it down, how to put it back . . . maybe they just need more strategies that they know they can go to, to do it.

Andrea said that there were too many "odd ways" students used to approach the problem, and she then mentioned that they might need more strategies. She hoped to have productive and efficient student discussions but did not know how to facilitate them. Her talk in this meeting was primarily a dialogue with another person (e.g., Amy or Aki) and the conversation did not include other teachers. As such, other teachers also carried on conversations without each other's input and moved from one topic to the next rather quickly.

Teacher talk in the middle of lesson study. The main goal of Meeting 5 was to discuss lesson details. The teachers talked about mathematics problems they planned to use in their lessons and tried different numbers in the problems to anticipate possible student responses.

Andrea: I think one thing that's neat about the number lines, though, is maybe (students) will see that they can use the addition to do the subtraction by counting up.

Aki: Especially since the problem kind of invites that.

Sophia: . . . challenge them to try different strategies that we come up with the first two lessons . . . I think the first lessons would be just talking about strategies, and then starting to do some of the problems.

Rosemary: Well, I don't know. I guess I feel like somehow I have to model it, you know, so that they get the concept of it. [I may say something like] *And I know you do it different ways, but this time I'd like you to try to solve it on a number line*.

Amy: Or [you may also say] *I'd like you to try it two different ways. And one of them* . . .

Rosemary: . . . *has to be the number line*. . . . It is to give them a choice?

Amy: It's to give them a choice. And also then they're certain . . .

Andrea: . . . it would be really neat just to see how the number line differs for each type of subtraction problem . . . we've done a little bit, but . . . it was sort of an idea of how to use it, but I don't think most of them use it on their own, correctly at least.

Aki: So this lesson is going to really help solidify their understanding.

Andrea: Yeah.

Sophia and Rosemary primarily focused on discussing teaching in this meeting, whereas Andrea was making connections to student thinking and learning. Teachers asked questions and shared their concerns about how to encourage their students to use number-line representations in the lessons.

In general, when one of the teachers in a lesson-study group teaches a research lesson, it is common for the teacher to have a preference for her own teaching style and approaches. It is important for the teachers to discuss in the group the reasons and rationale behind particular teaching approaches brought to the lesson, so that members accept and understand the teacher's approaches and have general agreement on the lesson's steps. This type of discussion can help teachers both articulate their own ideas and understand what others think about teaching. In this group, each teacher had a different comfort level with respect to requiring (or suggesting) a new representation use for students. Andrea's statements indicate that she was beginning to realize the connections between representation uses and subtraction problems, and with appropriate problems and guidance, her students were likely to use certain representations. She believed it would be interesting to see how her students might use different number-line representations. Although her talk focused primarily on student learning, she was also taking part in the discussion in which other teachers talked about teaching.

Teacher talk toward the end of lesson study. In Meeting 6, the teachers discussed the details of the lessons. As Figure 4 shows, in this meeting all of the teachers focused their talk on teaching, because they were getting closer to teaching the lessons. The teachers brought written lesson plans for their grade levels and asked questions to get feedback from one another. At the beginning of the meeting, Andrea shared her current concern:

Andrea: I'm worried that they [students] are going to say something and I won't know how to represent it. You know, anything with compensation,[6] I kind of have to think about it like, "Do we shift everything over on the number line?"

Aki: So with addition . . . if you add one, another one has to be subtracted, right?

Andrea: Yeah.

Aki: But with subtraction you have to kind of move together [on a number line].

Sophia: . . . [for a situation like that] I actually let them see me get frustrated.

Aki: That's good . . . you can ask them . . .

Sophia: I know. *We need some time to figure it out*.

Andrea: Yeah.

Aki: I think that's a great idea. Because that (compensation method) is a hard one. You know in your mind how addition and subtraction work, but knowing how to represent it on a number line can be a different thing.

Whereas Andrea's question was specific to how to represent the compensation strategy with a number line, her real concern was broader: How would she be able to represent all her students' thinking? Although she may not be able to anticipate and have a plan for all possible representations, talking with colleagues and clarifying some of the specific confusions seemed to be helpful as she prepared herself

[6]Examples for the compensation method and number line use are shown in Appendix B.

for teaching the lesson. Sophia mentioned that she would share her frustration in the thinking process with her students, and that presented an alternate image of the instruction. By having opportunities to discuss details of one lesson over multiple occasions, teachers shared what they were not certain about and asked questions. These shared experiences became the basis for their teaching, so that when they observed each other's lessons, they knew when to pay special attention in the lesson so that they could give additional feedback later.

Andrea's Learning: What Are Representations?

As was illustrated in the previous section, Andrea's lesson-study journey was long but supported by the group lesson-study path as well as by her colleagues' ideas. In Meeting 2, Andrea expressed her confusion about representations:

Andrea: . . . my students have too many strategies and I have a hard time teaching math . . . why are we teaching another strategy [number-line representation] now?

Andrea believed that students' multiple strategies should be recognized and valued, but in her mind, she did not make clear distinctions among these strategies or know how to prioritize them for different types of problems. As a consequence, facilitating student discussions when multiple strategies were presented was likely not easy for Andrea.

Andrea's learning was embedded in the lesson-study team's learning. For most of the lesson-study process, all three teachers used the word *representations* in their talk, but they did not appear to be clear about what they meant. In Meeting 5, the teachers continued to discuss their confusion:

Amy: What's interesting to me, though, in this and some others I've seen, they [the textbook] (are) not actually using the bar model[7] to do the work, to do the calculation. It's a representation . . .

Sophia: That's why I was confused! So how is it different than the algorithm in this case?

Andrea: I guess because we said one of the goals was "representation of subtraction"?

Amy: It's a representation. It's a different model.

Sophia: Like a visual representation would draw a picture versus . . .

Andrea: . . . Yeah.

Amy: It definitely shows the missing part. It's just that to find out how big the missing part is, you've still got to figure it out.

In teachers' discussions, it is apparent that they did not make a clear distinction between these different terms (representations, models, and algorithm) as they appear to consider representations used in the textbook as a way to calculate, to be

[7]A visual representation typical in Asian mathematics curricula that shows quantities using bar-like rectangles (also known as tape diagrams).

used as an algorithm. They found it confusing to see a representation used as a solution method (algorithm), while it could also model the problem situation. Representing problem situations and representing solution procedures in a problem-solving process do not always support one another (Fuson, 1992). Although scholarly definitions of these may be quite different from one another, the teachers' understanding of the terms was embedded in their own shared craft knowledge, and there is a gap between these two ways of knowing (scholarly and craft). The heavy instructional focus on quick solution can guide teachers' and students' attention to solving problems and away from actual problem contexts. The teachers needed support to make sense of the differences in their practice, in order to connect the scholarly with the craft.

In Meeting 6, the need to come to consensus about what they meant by *representations* pushed the teachers to focus and agree on how to pose a question in the lesson:

Aki: . . . And then the students will solve it [the problem] and make a representation. Do they know what a *representation* is? Have you used the word with them?

Rosemary: Not really.

Aki: What would you say? I mean this is a big question.

Rosemary: I guess I would say, to make it simple, *a picture*. … Or *show me.* I've said different things, *like you have to show me your work.*

Sophia: . . . I use the word *represent*. I've never used *representation*. . . . I use strategies in my classroom, so I'm looking at it as *strategies*.

Andrea: Well, remember that math meeting we had? It was [about] *strategies, models,* this and that. I'm afraid to use any of those words because I don't know if it's . . . are these really *models*, not s*trategies*?

Rosemary: . . . What if I said, "*What are some ways you could show what's happening?*"

Andrea: I think that's how I usually speak.

Rosemary: . . . Okay. *What are some ways you can show me what's happening?*

Sophia: Yeah, *what's happening* . . .

At the end of the discussion, they agreed to use a phrase, *show what's happening*, in their lessons when asking students to represent their ideas, because they felt comfortable with the phrase. Starting with considering representations as *showing what is happening*, the teachers, as a group, gradually came to think about how they would use unfamiliar terms in their own classrooms. Without the group context, a teacher might have designed a lesson without giving thorough consideration to how to communicate with students about what was meant by *representation.* The perceived need to come to a consensus pushed the teachers' thinking forward.

Andrea's understanding in action: *Research lesson*. Andrea planned four lessons for the instructional unit, in which she taught Lesson 3 as a research lesson (public lesson). In the first two lessons, she took the time to introduce the problem context and guided her students to use number-line representations for the problem situations. A travel context was chosen by the group, as they believed it was an ideal context to be shown with a linear representation. Following is the problem for Lesson 1:

> Elizabeth's family visited their grandmother, who lives 534 miles away. On the first day, they drove 319 miles. How much more did they have to drive on the second day?

Students showed the problem situation by first drawing pictures (with elaborate details), then drawing number-line representations to simplify the situation and focus on the mathematics in the problem. In Lesson 2, students were given a different problem:

> For spring break, Ms. Andrea and Ms. Susan each drove from Coastal Town to other cities. Ms. Susan drove 117 miles to the capital of California, Sacramento. Ms. Andrea drove 383 miles to Disneyland. How many more miles did Ms. Andrea drive?

Many students took the ideas from the previous lesson and drew similar number-line representations for the second problem, by placing two points, 117 and 383, on the line and finding the difference between the two (see Figure 6, student representation 1). There were only a few students who drew two number lines to show each person's travel distance and found the difference (see Figure 6, student representation 2).

Andrea was puzzled by the difference between the two representations and wanted to clarify the difference in the following lesson. As she started the lesson, she guided student discussion by reviewing what they had done in the previous lesson.

Andrea: First we started with a puzzle with Elizabeth's family traveling to visit Grandma. They stopped at the hotel for the night, and then kept on going, and what did we figure out there? What were they asking us? Jenna?

Jenna: . . . how many miles they traveled the second day.

Andrea: Yeah, how much more did they drive on the second day. What did you decide, Joe?

Joe: . . . Um, basically, you take the total number, and um, take away, um, the other number they show . . .

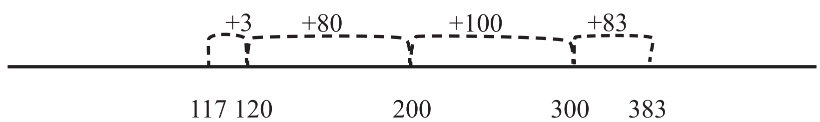
Andrea: Right, so this is a problem where they tell you the total number . . . you're saying that what you did was take it away . . . Now the next day, we had a problem where I went to Disneyland, Ms. Susan went to Sacramento, and we wanted to know how much further I drove than Ms. Susan. What type of problem did you figure out this was?

Jose: Comparing two numbers.

Andrea: Comparing. What shows us that this is a comparing problem, a comparison problem?

Problem: For spring break, Ms. Andrea and Ms. Susan each drove from Coastal Town to other cities. Ms. Susan drove 117 miles to the capitol of California, Sacramento. Ms. Andrea drove 383 miles to Disneyland. How many more miles did Ms. Andrea drive?

Student Representation 1



Student Representation 2

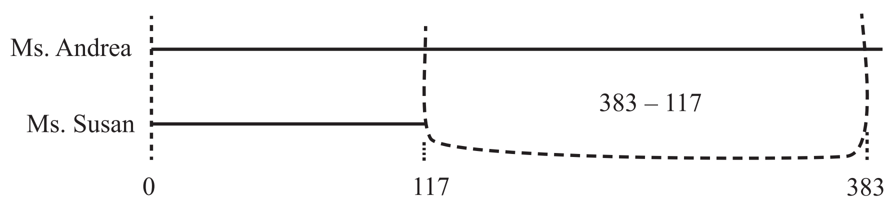


Figure 6. Different student number-line representations.

Raymond: 'Cause it's how much more did you drive?

Andrea: . . . how much more did I drive, and there's a picture of Mei that helped us with. She drew two number lines [see Figure 6, student representation 2], to show that I drove more. And we want to figure out this much [identify where the difference is shown in the representation]. And then, after the lesson, we talked about how most people did their number line as like a close-up view of this little piece, here [see Figure 6, student representation 1]. 'Cause almost everybody did their number line from a 117 to 383. And so, few people did from 0 to 383. So what most people were drawing was this little piece up here [shows the difference in student representation 2 again]. Okay?

Andrea discussed different problems and different representations that accompanied the problems, compared these representations, and made connections between different student representations focusing on the problem situation. She could identify how students paid attention differently to the problem, and thus was able to connect student thinking via representations. This was the primary goal of the lesson study.

In the final interview after lesson study, Andrea described her thinking about representations as follows:

> [In lesson study] I learned that there are many different ways to use the number line for problems. In some cases, we used two number lines to compare lengths. I think number-line representation uses are one of the ideal ways to really show how subtraction works. . . . It makes the connection to other methods, such as adding up and using friendly numbers. The number line shows why the other methods work.

Andrea's statement implies that she was now less confused about differences between different number-line representations. She considered representations as something to show a mathematics concept, make connections between methods, and a tool for students to discuss their ideas. She also learned how different representations are used appropriately for different types of problems, which should become a basis for her facilitation of student ideas in the future. Her learning, indicated by the increase in the talk levels as well as by her aforementioned statement, emerged in the multiple learning opportunities in the lesson-study process. Meeting discussions, questions posed by peers, and facilitating student discussions in the lessons all required Andrea to think about how student learning could be guided by content and teaching, and thus fostered connections among these dimensions.

Challenges With Number-Line Representations

In showing quantities with number-line representations, students and/or teachers may think the numbers marked on the number line are discrete points in the continuing paths of the quantities instead of the quantities that precede the point (often beginning with 0). Although it can be an appropriate interpretation, for some problem contexts it was important for students to consider the distance from the origin (point 0). For example, in student representation 1 in Figure 6, the distance that precedes 117 is not shown in the representation, and the number 117 indicates a point on the line. Representation 2 in the figure more accurately shows the distance from 0 that is in the problem situation. The difference between these two approaches is what Ng and Lee (2009) call the difference between a problem-solving heuristic (student representation 2) and students' tendency to use representations as algorithms (student representation 1). Andrea, in her teaching, helped make connections between these two approaches using students' own work as previously described.

Andrea was clearly on her own learning path toward better understanding of how to use representations in her teaching to support student learning. Her own emerging understanding was reflected in her students' emerging approaches to solving problems. In discussing different representations, students first focused on finding one correct representation over others and did not make connections among representations while considering them as representing the same problem. It is important to help teachers (and students) understand the difference between using a diagram as a solution method and as a situation representation, because merely using the diagram to solve a problem introduces yet another "strategy" for students, about which the teachers voiced concern at an earlier point in their lesson-study meeting. Using the diagram to support student understanding of the problem situation requires careful representation of the situation, which can lead to a solution method. The representation can meaningfully guide a solution method. By relating the common sections of the situation representation and the solution method, as in the previously described lesson, Andrea and the students together began to understand how the solution method could be drawn from the representation of the situation by the number line. The distinction between the situation representation and solution method, however, still appeared to be fairly vague at the end of the teaching of the lesson for both students and the teacher.

It becomes even more crucial for curriculum writers and professional developers to provide support for understanding and appropriate use of representations for teachers and students who may not have had many experiences with or deep understanding of certain representations. The teacher's manual that accompanied the textbook used for this lesson study, *Investigations in Number, Data, and Space* (Pearson Education, 2008), models problem situations with tape diagrams that embed quantities to show the relationship among the quantities in the problem. It was the teachers' collective decision to focus exclusively on number line representations in their lessons and not to use tape diagrams. The simultaneous use of tape diagrams and number lines (as originally outlined in the curriculum), however, could have made the connections between the problem situation and solution methods clearer, as tape diagrams show quantities in chunks, whereas number lines can show the discrete points in the path of quantities.[8]

It is important to expose, recognize, discuss, and understand these issues so that teachers (and their students) can learn about them. Fuson (1992) discusses the tension between representing problem situations and representing solution procedures in mathematics teaching, as the two modes do not always support one another. Because students can lack problem-type schemata for situations, many consider numbers in the problem to be central in interpreting the problem, to the extent that they see only numbers in the problem and manipulate them to find the answer without considering the problem setting (Fuson & Willis, 1986; Ishida, 1998; Lesh & Harel, 2003; Thompson & Hendrickson, 1983). The heavy focus on quick solutions can guide students' attention away from the problem contexts. In classrooms, the distinction between representing problem situations and representing mathematical solutions is often not discussed, and students (and teachers) often use one but not the other, successfully or unsuccessfully. We argue that both teaching and research will benefit if these two formats are clearly differentiated. As teachers understand the differences between the two formats, they could teach students mathematical problem solving in two steps, first to represent mathematical relationships, and then to solve problems using the relationships. By doing so, they could highlight mathematics content in the problem-solving process. Research could also investigate the different challenges in using representations in these two formats and separately make informed suggestions for instructional and curriculum design.

In lesson study, teachers' partial understanding often becomes visible, and in safe, collaborative settings it is possible for teachers to discuss and make sense of the challenges together, seek necessary information, and develop new and shared knowledge. Lesson study requires teachers to discuss openly and to come to consensus about aspects of a lesson they plan, thus exposing their incomplete understanding at times in a nonthreatening atmosphere. These professional learning

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[8]See Appendix C.

opportunities are rare in everyday teaching, during which the demands of teaching may cause teachers to feel as if they must move along quickly simply to cover content.

TEACHER-TALK PATHS

We have described how lesson study supported a group-learning path with this elementary lesson-study group, moving from a superficial consideration of student learning to content, teaching, and ultimately to a more purposeful consideration of student learning. The example illustrated how Andrea developed and used one kind of mathematics pedagogy, showing mathematics content and student thinking through representations. The teachers' discussion by the end of lesson study had become more connected to their students' learning (Level 4 in Figure 2) in the lesson and to their specific lesson learning goals. This lesson study guided teachers' work to focus on one or more of the three dimensions in Figure 1, moving it from student learning, to content, to teaching, and back to student learning, as shown in Figure 3.

Individual teachers took slightly different talk paths in the lesson study. In some meetings, all teacher talk was embedded in the group-talk path, whereas in other meetings, individual teacher talk split into different dimensions and/or outside the group path. The teachers' backgrounds and interests seemed to have influenced the different dimensions in their talk, which in turn affected their individual talk paths (see Figure 4). Although individual teachers may have veered outside the group-talk path, the group context centered their discussions and integrated and wove the individual paths together.

The lesson-study context helped facilitate the interactions of three dimensions in teacher talk. The teachers extended each other's ideas by asking questions and clarifying concepts. The discussion helped them make connections and develop their knowledge of teaching. This extends the previous study by van Es and Sherin (2008), in which the researchers identified different teacher-learning paths in a video club. In our study, the teachers also took different individual paths (as shown in Figure 4), and since they planned lessons collaboratively, they needed to consider and discuss each other's ideas in meetings and arrive at common teaching approaches that they could all use in the lessons. As examples in this article suggested, they examined their teaching (e.g., how to present a problem to students) in ways in which they might not have if they had worked independently. This process was uniquely shaped by the differences among these particular teachers, and it seemed to have helped to guide their learning in the group, as individual-talk and group-talk paths interacted. Such integrations of ideas could not have happened if teachers had worked in isolation.

When different dimensions of the triangle in Figure 1 interact in practice, teachers make sense of their teaching (Lampert, 2001). In the current case, different teachers brought different ideas to the collaborative setting, which focused on one or more dimensions of the model, and these dimensions interacted with one another in the context of the group to support teacher learning. Although the contributions of individuals can be unpredictable, this lesson study framed the interactions by moving the teachers through the three dimensions. As was shown in Figure 3, teachers were guided in the group-talk path structured by lesson study. Individual teachers' interests and experiences determined how individual paths varied from the group path and from other teachers' paths, and increases in the levels of their talk, collectively and individually, were evidence of learning. For example, Figure 3 shows that in Meeting 1 teachers' collective Level 1 talk dominated 51% of all talk, and but Level 1 talk decreased to 27% in Meeting 8. Simultaneously, Level 4 talk increased from 14% to 35%. All three teachers started with Level 1 of the student-learning dimension talk in Meeting 1, and after taking different talk paths, all reached Level 4 of the student-learning dimension by Meeting 8 (see Figure 4). It is in this process that group and individual paths interacted to support the teachers' learning. A contributing factor to this learning was the fact that teachers' discussions were consistently focused on and redirected to making connections among the dimensions as the lesson-study process unfolded. Effective facilitation in lesson study, supported by conceptually sound curriculum and teaching materials, was crucial in maximizing teacher learning in the process.

While the teacher talk developed and connected different aspects of teaching, the teachers also learned about mathematics pedagogy. The teachers in this study began to distinguish the differences between using representations to show the problem situations and as more general solution methods. As the case of Andrea showed, her mathematics pedagogy now includes the use of representations as a tool to show mathematics concepts and student thinking. This knowledge was developed in the course of her own professional investigation and not learned externally (e.g., in a single-day workshop), and Andrea's teaching, as well as her interview statement, evidenced this learning. Because Andrea was a relatively new teacher when the study was conducted (she had taught for 2 years prior to the study year), it is possible that her learning is not typical of all practicing teachers. Future studies can investigate the learning paths of more experienced teachers to determine how their connection-building processes may be different, given their teaching experiences and knowledge.

CONCLUSIONS AND IMPLICATIONS

Connecting Craft Knowledge of Teaching and Scholarly Knowledge of Teaching

The teachers who participated in this study developed new mathematics pedagogy and knowledge for teaching: how visual representations connect student thinking and mathematics content. This was evident in the increasing levels of all three teachers' talk over time, and in the case of Andrea, both in how she highlighted connections between different representations in her teaching and in her reflection statement.

This craft knowledge was developed collaboratively, influenced by the varied interests and experiences of the individual teachers, as evidenced by the ways they attended to different aspects of the discussion. By planning lessons, teaching them, and experiencing student learning in the lessons that confirmed (or refuted) their predictions about student thinking during the 6-month collaborative investigation, the teachers critically reexamined their everyday teaching. The shared investigation prepared teachers to focus on student learning when they observed each other's research lessons. When students responded to the problems in ways outlined by curriculum materials but that teachers had initially found challenging to envision, seeing these student responses in the lessons seemed to make the teachers reconsider what students were learning. When Andrea's students used double number line representations to show a comparison problem, it helped the teachers rethink the different ways the curriculum materials originally showed number lines. In this way, the teachers considered curriculum materials more deeply in lesson-planning processes.

The reciprocal support among group members, as well as the commitment they each made to teach public lessons, pushed these teachers to take a risk in teaching that was different from their usual style. This was evident when the public research lesson day approached and the teachers needed to come to agreement about how to teach the lessons. All three teachers expressed some degree of uncertainty in carrying out the lessons, as shown in a prior example (how to pose a problem, how to help students represent problem situations using number lines). In Andrea's case, she planned and taught lessons that drew largely on students sharing solution methods, although she expressed concerns that she was not sure how to facilitate student discussion. Collaboration in the lesson study helped ground the teachers' emerging knowledge in the practice they shared.

The facilitator shared scholarly knowledge only when the teachers expressed an interest in learning more and requested relevant information. The need for the teachers to ask for such information was in the lesson-study cycle; thus, as the facilitator guided the teachers through the process, the questions arose naturally. For example, as the teachers discussed their own students' methods for solving multidigit subtraction in Meetings 1 and 2, the facilitator asked the teachers whether they could see the connections across methods and across grade levels. In considering this question, the teachers asked why some methods were used in lower levels but not in upper levels, prompting the facilitator to bring summaries of research studies that investigated student learning of multidigit operations (e.g., Carpenter et al., 1998). In this manner, the scholarly knowledge shared by the facilitator was connected to the teachers' questions and needs. This process was not purely organic, as the lesson-study structure helped create the need for teachers to ask certain questions and to look for resources and support. However, the teachers' interests were always considered to be central to their professional investigation, in order to help them feel ownership of the knowledge-generation process.

In Japan, where lesson study originated, the distance between research and practice, or between scholarly knowledge and craft knowledge, is smaller than in the United States. It is common for practicing teachers to conduct their own research and publish books and articles in teacher journals to develop their individual and collective craft knowledge (Lewis, 2010; Lewis & Tsuchida, 1998; Stigler &

Hiebert, 1999). In general, knowledge is valued more when it can be used in practice (Suzuki, 2002), thus the teaching community is considered integral to the research community. The knowledge and materials developed by the teachers in this study, and in other lesson-study groups, can and should inform the field of mathematics education. Teachers who make their materials accessible to others and share their experiences in lesson study take the initial steps in meaningfully bridging scholarly and craft knowledge. Because of the differences between Japan and the United States in terms of teachers' familiarity with and investment in lesson study, careful and long-term investigation of U.S. lesson-study adoption processes will be important.

Interactions Between Group and Individual Learning in a Shared Learning Environment

In any learning setting, a learner (teacher or student) develops meanings of experiences based on personal interests, background, beliefs, and prior understanding. Some learning settings are based on the assumption that learning should be uniform across individuals, and the experiences are often strictly dictated and constrained. These controlled environments can lack authentic meaning, as individuals are not encouraged to make their own connections with the concept at hand or to construct individual understandings. In the exploratory learning environment, by contrast, differences among learners are expected and welcomed. The diversity of thinking provides fuel for the learning. Different learning paths that individuals take interact with one another in the context, and the interaction allows for a broader learning range than the original instructional path may have afforded. In collaborative discussions, different members push and extend each other in and out of the group-learning path, and this movement helps everyone involved in the context to understand the topic better, because one is likely to push back when he or she feels the discussion digresses too far (Murata, in press).

Learning is a social endeavor when multiple individuals' learning trajectories are meaningfully embedded in the group's path. In the case illustrated in this article, teachers learned together despite different foci on the dimensions they brought to the context, shown with the talk paths. We argue that similar group–individual interactions can be possible in other learning settings. Most learning happens in social settings (e.g., schools, workplaces) in the presence of others, and it is important to understand the dynamics created in interactions of ideas in such settings. Lesson study holds the potential to transform teacher learning from the transmission of knowledge from an "expert" to the teacher, to a journey in which the teacher's ideas are coordinated, with different ideas interacting and shaping teacher learning, guiding them toward a purposeful learning goal and destination.

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Accepted May 25, 2012

APPENDIX A

Unit Plans from *Investigations in Number, Data, and Space*
(Pearson Education, 2008)
Grade 4, Unit 5, Landmarks and Large Numbers: Addition, Subtraction, and the Number System Investigation 4: Subtraction

Session	Title of the session	Goals
4.1	Representing Subtraction Problems	Students represent and solve subtraction problems in a variety of contexts: missing part, comparison, and removal.
4.2	Strategies for Subtraction	Students discuss their subtraction strategies and how to represent them on a number line. They practice and are assessed on solving subtraction problems and on creating story contexts for subtraction.
4.3	Assessment: Numbers to 10,000	Students describe and name subtraction strategies, according to how each strategy starts. They continue practicing adding and subtracting numbers to 10,000 and adding and subtracting multiples of 10, 100, and 1,000. They are also assessed on fluency with numbers up to 10,000.
4.4	Do I Add or Subtract?	Students examine ways to start subtraction problems, including changing one number and adjusting. They use story contexts and representations to visualize how changing the amount subtracted affects the differences.
4.5	Solving Addition and Subtraction Problems	Students practice solving addition and subtraction problems with 3- and 4-digit numbers in the context of money and distance during Math Workshop.
4.6	Solving Addition and Subtraction Problems, continued	Students continue to practice solving addition and subtraction problems with 3- and 4-digit numbers. They discuss how to solve a problem that involves more than one step and more than one operation.
4.7	End-of-Unit Assessment	Students solve four problems to assess their understanding of addition, subtraction, and the structure of the number system to 10,000.

Note. This is the summary of the original unit plan from the curriculum. The lesson-study team used and modified Sessions 4.1 and 4.2 for their teaching.

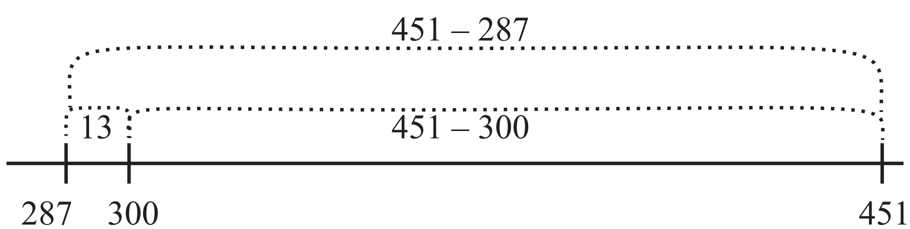
APPENDIX B

Summary of Description of "Changing the Numbers" Strategy from *Investigations in Number, Data, and Space* (Pearson Education, 2008)

In the original teacher's manual, the Changing the Numbers strategy is described as a group of strategies in which students change one of both of the numbers to what they often call "landmark" or "friendly" numbers. For example, a student may approach 451 – 287 as:

$$451 - 300 = 151$$
$$151 + 13 = 164,$$

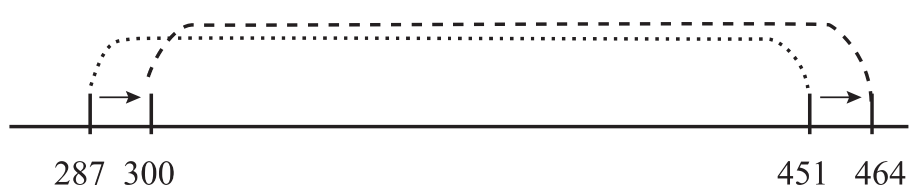
for which the original subtrahend, 287, is changed to 300, and the difference between the numbers (287 and 300), 13, is added at the end of the process.



The problem also can be solved by changing both numbers:

$$287 + 13 = 300$$
$$451 + 13 = 464$$
$$451 - 287 = 464 - 300 = 164$$

The student added 13 to both minuend and subtrahend to make it easier to subtract by using 300 as a friendly number.

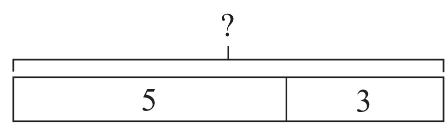


APPENDIX C

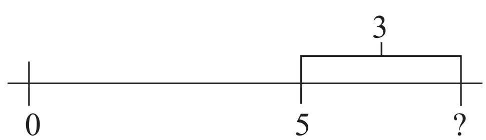
How Quantities Are Shown With Tape Diagram and Number-Line Representations

Problem: Toby biked 5 miles yesterday. He plans to bike 3 more miles today. How many miles would he bike by the end of today?

Tape-diagram representation:



Number-line representation:



By placing tape-diagram and number-line representations together, they help show the quantities on the number line. In the example above, while the 5 in the number-line representation indicates the specific point on the line, the 5 in the tape diagram above shows the quantity from 0. Thus, the instruction using these two representations together can help students see the quantities implied with the number line.